Classifying the nonsingular intersection curve of two quadric surfaces

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Abstract

We present new results on classifying the morphology of the nonsingular intersection curve of two quadrics by studying the roots of the characteristic equation, or the discriminant, of the pencil spanned by the two quadrics. The morphology of a nonsingular algebraic curve means the structural (or topological) information about the curve, such as the number of disjoint connected components of the curve in $\mathbb{PR}^3$ (the 3D real projective space), and whether a particular component is a compact set in any affine realization of $\mathbb{PR}^3$. For example, we show that two quadrics intersect along a nonsingular space quartic curve in $\mathbb{PR}^3$ with one connected component if and only if their characteristic equation has two distinct real roots and a pair of complex conjugate roots. Since the number of the real roots of the characteristic equation can be counted robustly with exact arithmetic, our results can be used to obtain structural information reliably before computing the parameterization of the intersection curve; thus errors in the subsequent computation that is most likely done using floating point arithmetic will not lead to erroneous topological classification of the intersection curve. The key technique used to prove our results is to reduce two quadrics into simple forms using a projective transformation, a technique equivalent to the simultaneous block diagonalization of two real symmetric matrices, a topic that has been studied in matrix algebra.

Key words: Intersection Curves; Quadrics; Block Diagonalization.

1. Introduction

Computing the intersection curve of two quadric surfaces is important in CAGD applications and thus remains an active research topic. We are concerned with real quadric surfaces in 3D real projective space, $\mathbb{PR}^3$, and their real intersection curve. The intersection curve of two quadrics will be referred to as QSIC. The intersection curve of two quadrics can be singular or nonsingular. A singular QSIC can be reducible or irreducible; in the former case the QSIC consists of some lower degree curves whose degrees sum to four, and in the latter case the QSIC has exactly one real singular point. A nonsingular QSIC has zero, one, or two disjoint connected components in $\mathbb{PR}^3$. The goal of this paper is to give some algebraic conditions for telling the number of components of a nonsingular QSIC, plus some other structural information about the QSIC.

The literature on the intersection of two quadrics abounds, including both classic results in algebraic geometry and modern ones in computer aided geometric design. The classic work usually assumes the setting of complex projective space in favor of its algebraic closedness [1, 12]; thus the classic results cannot be applied directly, without refinement or extension, to the setting of real projective space assumed in most CAD and computer graphics applications. For example, the real roots and imaginary roots of the discriminant of a quadric pencil are not distinguished in the Segre characteristic, and only the multiplicity of a root is of interest [1]. On the other hand, efficient and accurate methods have been developed in CAGD for computing the QSIC in 3D real projective or Euclidean space. An elegant method for tracing a QSIC is presented by Levin [6]. Robust parsing of singular QSIC based on the factorization of a degree four planar curve using exact arithmetic is studied by Farouki, Neff and O'Connor [3]. Subsequent work attempting to refine or improve Levin's method can be found in [10, 14, 4].

When the input quadrics are assumed to be special, such as spheres, circular right cones or cylinders, called *natural quadrics*, there are many existing methods that exploit geometric observations to yield robust methods for computing the QSIC [7, 8, 11].

A problem with existing methods for computing the

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QSIC is that the topological information about the QSIC, such as the number of disjoint connected components, is not computed when the QSIC is nonsingular. Although it is possible that some of these methods, such as Levin's method, could be further analyzed to derive required topological information from the parameterization of the QSIC, we take a different approach here. We show that, with minimal computation using exact arithmetic, the number of real roots of the characteristic equation of any two quadrics can be counted to determine the morphology of their QSIC, supposing that the QSIC is nonsingular. (Note that singular QSICs can be detected and precluded by the fact that the characteristic equation of the two quadrics has a multiple root [3].) For example, we will show that two quadrics intersect along a nonsingular space quartic curve in $\mathbb{PR}^3$ with one connected component if and only if the characteristic equation has two distinct real roots and a pair of complex conjugate roots. The significance of our results lies in that the morphology of a nonsingular QSIC can be determined before engaging a lengthy floating-point computational procedure to obtain the parameterization of the QSIC, thus avoiding the danger that the errors incurred by the subsequent computation might lead to an incorrect topological classification of the QSIC.

The main results are summarized as follows. Let two quadrics be defined by $\mathcal{A}$: $X^TAX = 0$ and $\mathcal{B}$: $X^TBX = 0$, where $X = (x, y, z, w)^T$ are the projective coordinates and A, B are 4×4 real symmetric matrices. The *characteristic polynomial* of $\mathcal{A}$ and $\mathcal{B}$ is defined to be $\mathrm{f}(\lambda) = \det(\lambda A - B)$, and $\mathrm{f}(\lambda) = 0$ is called the *characteristic equation*; $\mathrm{f}(\lambda) = \det(\lambda A - B)$ is also called the *discriminant* of the pencil formed by $\mathcal{A}$ and $\mathcal{B}$. Suppose that the QSIC of $\mathcal{A}$ and $\mathcal{B}$ is nonsingular, i.e., all the four roots of $\mathrm{f}(\lambda) = 0$ are distinct. Then it can be shown that:

1. The QSIC of $\mathcal{A}$ and $\mathcal{B}$ has either two *affinely finite* components or no real points in $\mathbb{PR}^3$ if and only if $\mathrm{f}(\lambda) = 0$ has four distinct real roots.

2. The QSIC has one *affinely finite* component in $\mathbb{PR}^3$ if and only if $\mathrm{f}(\lambda) = 0$ has two distinct real roots and a pair of complex conjugate roots.

3. The QSIC has two *affinely infinite* components in $\mathbb{PR}^3$ if and only if $\mathrm{f}(\lambda) = 0$ has two distinct pairs of complex conjugate roots.

The notion of the affine finiteness of a component of a curve in $\mathbb{PR}^3$ will be defined later in section 2. Briefly speaking, a component is called *affinely infinite* if it is intersected by every plane in $\mathbb{PR}^3$; otherwise, it is called *affinely finite*. Clearly, whether a curve component is affinely finite or not is a projective property. To our knowledge, we are the first to consider this property of the QSIC and to relate it to the characteristic equation of the two quadrics.

The key technique used in the proof of the above results is to use a projective transformation to reduce $\mathcal{A}$ and $\mathcal{B}$ simultaneously into some simple forms which can be analyzed more easily. This technique is similar to and inspired by the work of Ocken, Schwartz and Sharir [9] who use it to compute a parameterization of the QSIC. There are the following differences in the formulations, purposes, and results of our work and the work in [9]: (1) In [9] the purpose is to present procedures for computing a parameterization of a QSIC, instead of finding a relation between the roots of the characteristic equation and the morphology of the QSIC, which is the goal of our work in the present paper; (2) The procedures presented in [9] for quadratic form reduction are lengthy and hard to follow, and the output cases of these procedures are not rigorously analyzed; as a result, some cases that do not arise are unnecessarily considered. In contrast, for quadratic form reduction we invoke from linear algebra standard results on simultaneous block diagonalization of two real symmetric matrices; (3) Some conditions and different topological configurations of the QSIC are not distinguished in [9]; for example, the case where the characterization equation has two real roots and the case where the characterization equation has four real roots are considered as one case there.

The remainder of this paper is organized as follows. In Section 2 we will give preliminaries, including the results on simultaneous block diagonalization of two real symmetric matrices, and some properties of a quadric pencil and its base curve (i.e., the QSIC). In Section 3, three sufficient and necessary conditions for three different morphologies of a QSIC are proved. The paper is concluded in Section 4 with some open problems.

2. Preliminaries

First we cite a result on simultaneous block diagonalization of two real symmetric matrices [13], which will be used heavily later in this paper.

Definition 1: *Let A and B be two real symmetric matrices with A being nonsingular. Then A and B are called a* nonsingular pair of r.s. matrices.

Definition 2: *A square matrix of the form*

$$M = \begin{pmatrix} \lambda & e & & \\ & \cdot & \cdot & \\ & & \cdot & e \\ & & & \lambda \end{pmatrix}_{k \times k}$$

is called a Jordan block of type I *if $\lambda \in R$ and $e = 1$ for $k \geq 2$ or if $M = (\lambda)$ with $\lambda \in R$ for $k = 1$; M is called a* Jordan block of type II *if*

$$\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad a, b \in R, \; b \neq 0$$

and

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for $k \geq 4$ or if

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for $k = 2$, with $a, b \in R$, $b \neq 0$.

Recall that two square matrices $P_{n\times n}$ and $Q_{n\times n}$ are congruent if there exists a nonsingular matrix $V_{n\times n}$ such that $P = V^TQV$.

Theorem 1 (Canonical pair form theorem) [13]: *Let A and B be a nonsingular pair of r.s. matrices. Suppose that $A^{-1}B$ has real Jordan normal form $\mathrm{diag}(J_1, ...J_r, J_{r+1}, ...J_m)$, where $J_1, ...J_r$ are Jordan blocks of type I corresponding to the real eigenvalues of $A^{-1}B$ and $J_{r+1}, ...J_m$ are Jordan blocks of type II corresponding to the complex eigenvalues of $A^{-1}B$. Then A and B are simultaneously congruent by a real congruence transformation to $\mathrm{diag}(\varepsilon_1E_1, ...\varepsilon_rE_r, E_{r+1}, ...E_m)$ and $diag(\varepsilon_1E_1J_1, ...\varepsilon_rE_rJ_r, E_{r+1}J_{r+1}, ...E_mJ_m)$, respectively, where $\varepsilon_i = \pm1$ and E_i are of the form*

$$\begin{pmatrix} 0 & & 1 \\ & \cdot & \\ \cdot & & \\ 1 & & 0 \end{pmatrix}$$

of the same size as J_i for $i = 1, 2, .., m$.

Next we state some properties about a quadric pencil. Let $\mathcal{A}$: $X^TAX = 0$ and $\mathcal{B}$: $X^TBX = 0$ be two distinct quadrics, where A and B are 4×4 real symmetric matrices. We call a quadric $\mathcal{A}$: $X^TAX = 0$ a nonsingular quadric if the matrix A is nonsingular. Then $X^T(\lambda A - B)X = 0$, is called the pencil of quadrics formed by $\mathcal{A}$ and $\mathcal{B}$. It is easy to see that the QSIC of two distinct quadrics $\mathcal{A}$ and $\mathcal{B}$ is the same as the QSIC of any two distinct quadrics in the pencil $X^T(\lambda A - B)X = 0$; thus the QSIC of $\mathcal{A}$ and $\mathcal{B}$ is also called the *base curve* of their pencil.

Given two distinct quadrics $\mathcal{A}$: $X^TAX = 0$ and $\mathcal{B}$: $X^TBX = 0$, another pair of quadrics $\mathcal{A}'$: $X^TA'X = 0$ and $\mathcal{B}'$: $X^TB'X = 0$ are called the *projectively equivalent forms* of $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{PR}^3$, if A' and B' are congruent A and B, respectively, by the same congruence transformation. Since the topological properties of a QSIC that will be studied in this paper are invariant under real projective transformations, the QSIC of two quadrics has the same topological structure as the QSIC of any of their projectively equivalent forms in $\mathbb{PR}^3$.

Definition 3: *Let S be a set of points in $\mathbb{PR}^3$. If $S \cap L = \emptyset$ for some plane L in $\mathbb{PR}^3$, then S is said to be* affinely finite*; otherwise, S is said to be* affinely infinite.

One may convert $\mathbb{PR}^3$ into a 3D real affine space $\mathbb{AR}^3$ by choosing a plane L in $\mathbb{PR}^3$ as the plane at infinity; in this case, $\mathbb{AR}^3$ is called an *affine realization* of $\mathbb{PR}^3$. If a connected component of a QSIC is affinely finite, it is possible to find an affine realization $\mathbb{AR}^3$ of $\mathbb{PR}^3$ such that the component is bounded in $\mathbb{AR}^3$. If a connected component of a QSIC is affinely infinite, then it is unbounded in any affine realization of $\mathbb{PR}^3$. A component of a nonsingular QSIC may be affinely finite or affinely infinite. Clearly, affine finiteness is a projective property of a point set in $\mathbb{PR}^3$. This property will be used in classifying the morphology of a nonsingular QSIC.

3. Conditions for classifications

Since the characteristic equation $\mathrm{f}(\lambda) = 0$ is a quartic equation with real coefficients, it may have zero, two, or four real roots. In this section we will first prove three sufficient conditions (Theorems 2 to 4) for three different morphologies of a nonsingular QSIC, and then show that these conditions are also necessary (Theorem 5).

3.1. $\mathbf{f}(\lambda) = 0$ has four distinct real roots

Theorem 2: *Given two quadrics $\mathcal{A}$: $X^TAX = 0$ and $\mathcal{B}$: $X^TBX = 0$, if their characteristic equation $f(\lambda) = 0$ has four distinct real roots, then the $QSIC$ of $\mathcal{A}$ and $\mathcal{B}$ has either two affinely finite connected components or no real points in $\mathbb{PR}^3$.*

Proof: Let λ_i, $i = 1, 2, 3, 4$, be the four distinct real roots of $\mathrm{f}(\lambda) = 0$. Since the QSIC of $\mathcal{A}$ and $\mathcal{B}$ is nonsingular, all, except at most four, members of the pencil of $\mathcal{A}$ and $\mathcal{B}$ are nonsingular. Thus, without loss of generality, we may suppose that A is nonsingular; for otherwise, if A is singular, we may always choose another nonsingular quadric in the pencil to replace $\mathcal{A}$. Hence, we may suppose A and B form a nonsingular pair of r.s. matrices.

By Theorem 1, A and B can be simultaneously diagonalized by the congruence transformation $V = (V_1, V_2, V_3, V_4)$, where the V_i are the four real eigenvectors corresponding to the λ_i, $i = 1, 2, 3, 4$, with $J_i = (\lambda_i)$ and $E_i = (1)$. By a further congruent transformation to both V^TAV and V^TBV by the matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{|\tilde{A}_{1,1}|}} & & & \\ & \frac{1}{\sqrt{|\tilde{A}_{2,2}|}} & & \\ & & \frac{1}{\sqrt{|\tilde{A}_{3,3}|}} & \\ & & & \frac{1}{\sqrt{|\tilde{A}_{4,4}|}} \end{pmatrix},$$

where the $\tilde{A}_{i,i}$ are the diagonal elements of V^TAV, A and

B are reduced to the following respective congruent forms

$$A' = \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \varepsilon_3 & \\ & & & \varepsilon_4 \end{pmatrix},$$

$$B' = \begin{pmatrix} \varepsilon_1\lambda_1 & & & \\ & \varepsilon_2\lambda_2 & & \\ & & \varepsilon_3\lambda_3 & \\ & & & \varepsilon_4\lambda_4 \end{pmatrix},$$

where $\varepsilon_i = \pm 1, i = 1, 2, 3, 4$.

By setting B' to $B' - \lambda_1 A'$, which is another quadric in the pencil of $\mathcal{A}'$ and $\mathcal{B}'$, we get

$$B' = \begin{pmatrix} 0 & & & \\ & \varepsilon_2(\lambda_2 - \lambda_1) & & \\ & & \varepsilon_3(\lambda_3 - \lambda_1) & \\ & & & \varepsilon_4(\lambda_4 - \lambda_1) \end{pmatrix}.$$

Since $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are distinct, without loss of generality, we may suppose that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. It follows that $\lambda_2 - \lambda_1 > 0$, $\lambda_3 - \lambda_1 > 0$ and $\lambda_4 - \lambda_1 > 0$. Hence, the diagonal elements in matrix A' have the same signs with their corresponding elements in matrix B', except for $A'_{1,1} = \varepsilon_1$.

If ε_2, ε_3 and ε_4 have the same sign, $\mathcal{B}'$ is an imaginary quadric, and therefore the intersection of $\mathcal{A}'$ and $\mathcal{B}'$ has no real points.

If ε_2, ε_3 and ε_4 have different signs, the quadric $\mathcal{B}'$ is an elliptic cylinder or a hyperbolic cylinder parallel to the x-axis; in the latter case we may apply a further congruent transformation to make $\mathcal{B}'$ an elliptic cylinder parallel to the x-axis (with the same transformation also applied to A'). For the simplicity of notation, the resulting matrices are still denoted by A' and B'. Then $B'_{2,2}$ and $B'_{3,3}$ have the same sign, and the sign of $B'_{4,4}$ is different from that of $B'_{2,2}$ and $B'_{3,3}$. Hence, $A'_{2,2}$ and $A'_{3,3}$ have the same sign and the sign of $A'_{4,4}$ is different from that of $A'_{2,2}$ and $A'_{3,3}$. Then the quadric $\mathcal{A}' : X^T A' X = 0$ is either a unit sphere (when $A'_{1,1}$, $A'_{2,2}$ and $A'_{3,3}$ have the same signs) or a one-sheet hyperboloid with the x-axis as its central axis (when the sign of $A'_{1,1}$ is different from that of $A'_{2,2}$ and $A'_{3,3}$).

When $\mathcal{A}'$ is a unit sphere, its section with the y-z plane is a unit circle, while the section of the quadric $\mathcal{B}'$ with the y-z plane is an ellipse. Clearly, if one of the ellipse's semi-axes is smaller than 1, the QSIC of $\mathcal{A}'$ and $\mathcal{B}'$ has two oval branches; if both of the ellipse's semi-axes are greater than 1, $\mathcal{A}'$ and $\mathcal{B}'$ have no real intersection points. Note that none of the semi-axes can be equal to 1, since $\mathrm{f}(\lambda) = 0$ has no multiple roots.

When $\mathcal{A}'$ is a one-sheet hyperboloid, its section in the y-z plane is again a unit circle. The section of the quadric $\mathcal{B}'$ with y-z plane is still an ellipse. The ellipse and the unit circle cannot be tangential at any point, for otherwise the characteristic equation would have a multiple root, contradicting to that the QSIC is nonsingular. Then the ellipse either intersects the unit circle at four points, or contains the circle, or is contained in the circle. The QSIC has two oval branches in the former two cases, as shown in Figure 1, and the QSIC has no real points in the latter case.

Evidently, any component of the above QSIC of $\mathcal{A}'$ and $\mathcal{B}'$ in $\mathbb{PR}^3$, if exists, is affinely finite. Since the QSIC of $\mathcal{A}'$ and $\mathcal{B}'$ is projectively equivalent to the QSIC of $\mathcal{A}$ and $\mathcal{B}$, the proof is completed.

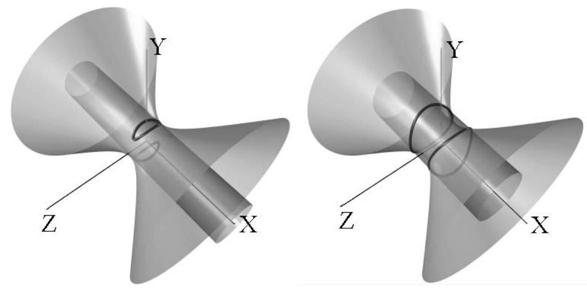

Figure 1. Two of the cases of an elliptic cylinder intersecting with a hyperboloid with one sheet.

3.2. $\mathbf{f}(\lambda) = 0$ has two distinct real roots and one pair of complex conjugate roots

Theorem 3: *Given two quadrics $\mathcal{A}$: $X^T A X = 0$ and $\mathcal{B}$: $X^T B X = 0$, if $f(\lambda) = 0$ has two distinct real roots and one pair of complex conjugate roots, the QSIC comprises one affinely finite component in $\mathbb{PR}^3$.*

Proof: Let λ_1, λ_2 denote the two distinct real roots and $\lambda_{3,4} = a \pm bi$ the pair of complex conjugate roots of $\mathrm{f}(\lambda) = 0$. As argued at the beginning of the proof of Theorem 2, we may suppose that A and B form a nonsingular pair of r.s. matrices.

Setting B to $(B - aA)/b$, which is a quadric in the pencil of $\mathcal{A}$ and $\mathcal{B}$, the pair of complex conjugate roots are mapped to $\pm i$. This does not change the topological properties of the QSIC of $\mathcal{A}$ and $\mathcal{B}$. By Theorem 1, A and B can be simultaneously block diagonalized by a congruence transformation into the following forms:

$$\begin{aligned} A' &= \begin{pmatrix} E_1 & & \\ & \varepsilon_1 & \\ & & \varepsilon_2 \end{pmatrix}, \\ B' &= \begin{pmatrix} E_1 J_1 & & \\ & \varepsilon_1\lambda_1 & \\ & & \varepsilon_2\lambda_2 \end{pmatrix} \end{aligned} \qquad (1)$$

where $\varepsilon_i = \pm 1$, $i = 1, 2$, and

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the following we need to consider two subcases: (1) $\lambda_1\lambda_2 \neq 0$ and (2) $\lambda_1\lambda_2 = 0$.

Subcase (1): By swapping λ_1 and λ_2, if necessary, we may suppose $\frac{\lambda_1}{\lambda_2} < 1$ in this case. By setting A' to $\lambda_1 A' - B'$, and then scaling the diagonal elements of B' into ± 1 by a further congruent transformation and using the same symbols A' and B' for the resulting matrices, we get A' and B' in the following forms:

$$A' = \begin{pmatrix} 1 & \lambda_1 & & \\ \lambda_1 & -1 & & \\ & & 0 & \\ & & & \varepsilon_2(\frac{\lambda_1}{\lambda_2} - 1) \end{pmatrix},$$

$$B' = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \varepsilon_1 & \\ & & & \varepsilon_2 \end{pmatrix}.$$

If $\varepsilon_2 = 1$, we swap $B'_{4,4}$ and $B'_{1,1}$, as well as $A'_{4,4}$ and $A'_{1,1}$, simultaneously by a congruence transformation to get

$$A' = \begin{pmatrix} (\frac{\lambda_1}{\lambda_2} - 1) & & & \\ & -1 & & \lambda_1 \\ & & 0 & \\ & \lambda_1 & & 1 \end{pmatrix},$$

$$B' = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon_1 & \\ & & & -1 \end{pmatrix}.$$

Or, if $\varepsilon_2 = -1$, we swap $B'_{4,4}$ and $B'_{2,2}$, as well as $A'_{4,4}$ and $A'_{2,2}$, simultaneously by a congruence transformation to get

$$A' = \begin{pmatrix} 1 & & & \lambda_1 \\ & (1 - \frac{\lambda_1}{\lambda_2}) & & \\ & & 0 & \\ \lambda_1 & & & -1 \end{pmatrix},$$

$$B' = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \varepsilon_1 & \\ & & & 1 \end{pmatrix}.$$

Hence, no matter $\varepsilon_2 = 1$ or $\varepsilon_2 = -1$, after a proper simultaneous congruent transformation, $\mathcal{B}'$ is a unit sphere or a one-sheet hyperboloid with the z-axis as its central axis.

Because $\frac{\lambda_1}{\lambda_2} < 1$, $A'_{1,1}$ and $A'_{2,2}$ have the same sign. Therefore $\mathcal{A}'$ is an elliptic cylinder parallel to the z-axis. Due to the symmetry of $\mathcal{B}'$ and $\mathcal{A}'$ about the x-y plane, we just need to analyze the relationship between the two conic sections that $\mathcal{A}'$ and $\mathcal{B}'$ have with the x-y plane.

The quadric $\mathcal{B}'$ intersects the x-y plane in the unit circle $x^2 + y^2 = 1$, and $\mathcal{A}'$ intersects the x-y in an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{(y-c)^2}{b^2} = 1$$

when $\varepsilon_2 = 1$, or

$$\frac{(x+c)^2}{b^2} + \frac{y^2}{a^2} = 1$$

when $\varepsilon_2 = -1$, where
$a = \sqrt{\frac{\lambda_2(1+\lambda_1^2)}{(\lambda_2-\lambda_1)}}$, $b = \sqrt{1+\lambda_1^2}$, and $c = \lambda_1$.
In both cases, the center of the ellipse shifts from the origin (along the x or y direction) by distance $|\lambda_1|$, and the length of the ellipse's semi-axis in the shift direction is $b = \sqrt{1+\lambda_1^2}$. Then it is straightforward to verify that one of the ellipse's extreme points of this axis is inside the unit circle, while the other is outside the unit circle. See Figure 2. This indicates that the QSIC of $\mathcal{A}'$ and $\mathcal{B}'$ has one component in $\mathbb{PR}^3$ (see Figure 3); this component is obviously affinely finite.

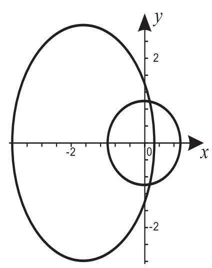

Figure 2. The cross-sections of an elliptic cylinder and a hyperboloid with one sheet in x-y plane.

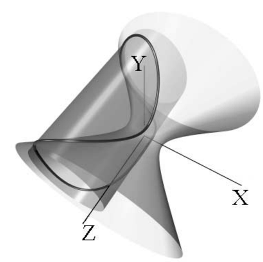

Figure 3. 3D illustration of the intersection between the two quadrics.

Subcase (2): We suppose that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then, according to Eqn. (1), A' and B' are reduced to the following forms:

$$A' = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \varepsilon_1 & \\ & & & \varepsilon_2 \end{pmatrix},$$

$$B' = \begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & 0 & \\ & & & \varepsilon_2\lambda_2 \end{pmatrix}.$$

Scaling the diagonal elements of B' into ± 1 (i.e., making $\varepsilon_2\lambda_2$ into ε_2) by a congruent transformation (with the same transformation also applied to A'), and then setting A' to $A' - \frac{1}{\lambda_2}B'$, we get A' and B' in the following forms:

$$A' = \begin{pmatrix} \frac{1}{\lambda_2} & 1 & & \\ 1 & \frac{-1}{\lambda_2} & & \\ & & \varepsilon_1 & \\ & & & 0 \end{pmatrix},$$

$$B' = \begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & 0 & \\ & & & \varepsilon_2 \end{pmatrix}.$$

When $\varepsilon_2 = 1$, we swap $B'_{4,4}$ and $B'_{1,1}$, as well as $A'_{4,4}$ and $A'_{1,1}$, by a simultaneous congruence transformation to obtain

$$A' = \begin{pmatrix} 0 & & & \\ & \frac{-1}{\lambda_2} & & 1 \\ & & \varepsilon_1 & \\ & 1 & & \frac{1}{\lambda_2} \end{pmatrix},$$

$$B' = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}.$$

Or, when $\varepsilon_2 = -1$, we swap $B'_{4,4}$ and $B'_{2,2}$, as well as $A'_{4,4}$ and $A'_{2,2}$, by a simultaneously congruence transformation to obtain

$$A' = \begin{pmatrix} \frac{1}{\lambda_2} & & & 1 \\ & 0 & & \\ & & \varepsilon_1 & \\ 1 & & & \frac{-1}{\lambda_2} \end{pmatrix},$$

$$B' = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

In both cases, $\mathcal{B}'$ is a circular cylinder with the z-axis as its central axis, and $\mathcal{A}'$ is either an elliptic cylinder or a hyperbolic cylinder, depending on the sign of ε_1 and λ_2, and $\mathcal{A}'$ is parallel to the y-axis. The equation of $\mathcal{A}'$ is

$$\frac{(y-c)^2}{a^2} \pm \frac{z^2}{b^2} = 1$$

when $\varepsilon_2 = 1$, or

$$\frac{(x+c)^2}{a^2} \pm \frac{z^2}{b^2} = 1$$

when $\varepsilon_2 = -1$, where
$a = \sqrt{1+\lambda_2^2}$, $b = \sqrt{\frac{1+\lambda_2^2}{|\lambda_2|}}$, $c = \lambda_2$.
The cylinder A' shifts by a distance $|\lambda_2|$ along the x-axis or the y-axis (depending on the sign of ε_2), and the length of its semi-axis in the shift direction is $\sqrt{1+\lambda_2^2}$. Then, it is

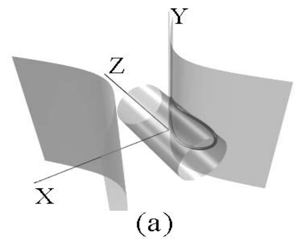

Figure 4. The intersection of a circular cylinder with a hyperbolic cylinder ((a)) or an elliptic cylinder ((b)).

straightforward to verify that the QSIC of the cylinders $\mathcal{A}'$ and $\mathcal{B}'$ comprises one component in $\mathbb{PR}^3$, which is affinely finite. See Figure 4. This completes the proof of Theorem 3.

3.3. f$(\lambda) = 0$ has two distinct pairs of complex conjugate roots

Theorem 4: *Given two quadrics $\mathcal{A}$: $X^TAX = 0$ and $\mathcal{B}$: $X^TBX = 0$, if $f(\lambda) = 0$ has two distinct pairs of complex conjugate roots, then the $QSIC$ of $\mathcal{A}$ and $\mathcal{B}$ comprises two affinely infinite components in $\mathbb{PR}^3$.*

Proof: Again, according to the argument in the proof of Theorem 2, we may assume that A and B form a nonsingular pair of real symmetric matrices. Suppose that the two distinct pairs of complex conjugate roots are $a \pm bi$ and $c \pm di$. Setting B to $(B - cA)/d$, we transform the roots $c \pm di$ into $\pm i$. By Theorem 1, A and B have the following simultaneous block diagonalization forms:

$$A' = \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix}, \quad B' = \begin{pmatrix} E_1 J_1 & \\ & E_2 J_2 \end{pmatrix};$$

here $E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ correspond to the roots $\pm i$, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J_2 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ correspond to the roots $a \pm bi$. Expanding A' and B', we get

$$A' = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -b & a \\ & & a & b \end{pmatrix}.$$

Clearly, $a \neq 0$ or $b \neq \pm 1$, since the pair of roots $a \pm bi$ are distinct from $\pm i$. Also, $b \neq 0$ since $a \pm bi$ are imaginary. Wlog, we may assume $b > 0$.

In the following we will derive a parameterization of the QSIC from which the topological information about the QSIC can be deduced. The quadric $\mathcal{A}'$ above, being a hyperbolic paraboloid, can be parameterized by

$$\mathbf{r}(u, v) = \mathbf{g}(u) + \mathbf{h}(u)v$$

where

$$\mathbf{g}(u) = \begin{pmatrix} -u \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{h}(u) = \begin{pmatrix} 0 \\ 1 \\ u \\ 0 \end{pmatrix}.$$

Substituting $\mathbf{r}(u, v)$ into $X^T B' X = 0$, we obtain

$$\mathbf{g}(u)^T B' \mathbf{g}(u) + 2\mathbf{g}(u)^T B' \mathbf{h}(u)v + \mathbf{h}(u)^T B' \mathbf{h}(u)v^2 = 0$$

or

$$v = \frac{-\mathbf{g}(u)^T B' \mathbf{h}(u) \pm \sqrt{s(u)}}{\mathbf{h}(u)^T B' \mathbf{h}(u)}, \qquad (2)$$

where

$$\begin{aligned} s(u) &= [\mathbf{g}(u)^T B' \mathbf{h}(u)]^2 \\ &\quad - [(\mathbf{g}(u)^T B' \mathbf{g}(u))(\mathbf{h}(u)^T B' \mathbf{h}(u))] \\ &= -bu^4 + (a^2 + b^2 + 1)u^2 - b. \end{aligned}$$

Substituting (2) into $\mathbf{r}(u, v)$ yields the following parameterization of the QSIC:

$$\begin{aligned} \mathbf{p}(u) &= [\mathbf{h}(u)^T B' \mathbf{h}(u)]\mathbf{g}(u) \\ &\quad - [\mathbf{g}(u)^T B' \mathbf{h}(u) \pm \sqrt{s(u)}]\mathbf{h}(u). \end{aligned}$$

Denote $\mathbf{p}(u) = (x(u), y(u), z(u), w(u))^T$, where

$$\begin{aligned} x(u) &= bu^3 - u, \\ y(u) &= -(au \pm \sqrt{s(u)}), \\ z(u) &= -u(au \pm \sqrt{s(u)}), \\ w(u) &= 1 - bu^2. \end{aligned}$$

For later use, define

$$\begin{aligned} \mathbf{q}_0(u) &= [\mathbf{h}(u)^T B' \mathbf{h}(u)]\mathbf{g}(u) \\ &\quad - [\mathbf{g}(u)^T B' \mathbf{h}(u) + \sqrt{s(u)}]\mathbf{h}(u), \end{aligned}$$

$$\begin{aligned} \mathbf{q}_1(u) &= [\mathbf{h}(u)^T B' \mathbf{h}(u)]\mathbf{g}(u) \\ &\quad - [\mathbf{g}(u)^T B' \mathbf{h}(u) - \sqrt{s(u)}]\mathbf{h}(u). \end{aligned}$$

Below we first show that $s(u) = 0$ always has four distinct real roots. The equation $s(u) = -bu^4 + (a^2 + b^2 + 1)u^2 - b = 0$ can be regarded as a quadratic equation in u^2, with discriminant

$$\begin{aligned} \Delta &= (a^2 + b^2 + 1)^2 - 4b^2 \\ &= a^2(a^2 + 2b^2 + 2) + (b^2 - 1)^2 > 0, \end{aligned}$$

since $a \neq 0$ or $b \neq \pm 1$. Thus there are two real solutions of u^2, given by

$$u^2 = \frac{(a^2 + b^2 + 1) \pm \sqrt{\Delta}}{2b}. \qquad (3)$$

Since $\Delta = (a^2 + b^2 + 1)^2 - 4b^2$ and $b \neq 0$, we have $(a^2 + b^2 + 1) > \sqrt{\Delta}$. Hence, the numerator in (3) is positive. Since $b > 0$ by assumption, the denominator in (3) is also positive. So we get two pairs of real solutions for u from (3), denoted by $\pm u_+$ and $\pm u_-$, with $u_+ > u_- > 0$, which constitute the four real roots of $s(u) = 0$. Denote

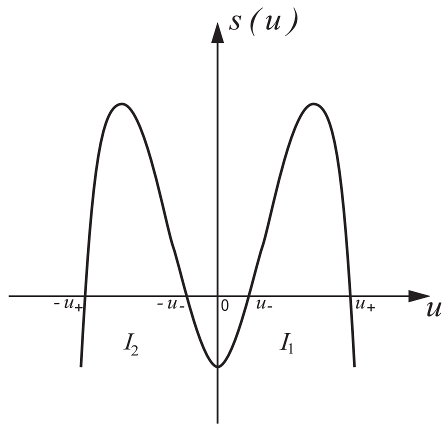

Figure 5. The graph of s(u) and its roots distribution.

the two intervals by $I_1 = (u_-, u_+)$, $I_2 = (-u_+, -u_-)$. Since $s(0) = -b < 0$, we have $s(u) > 0$ for $u \in I_1 \bigcup I_2$ and $s(u) \leq 0$ for the other values of u. See Figure 5 for the graph of $s(u)$. This indicates that the QSIC, defined by $\mathbf{p}(u)$, has two components, denoted by P_1 and P_2, corresponding to the two intervals I_1 and I_2, respectively: P_1 is defined by $\mathbf{p}(u)$ over I_1, and P_2 is defined by $\mathbf{p}(u)$ over I_2.

Now we show that the two components of the QSIC are affinely infinite. Since the two components have the same parametric expression (but over different intervals), we will only analyze the component P_1. Consider the QSIC, $\mathbf{p}(u)$, in the affine space $\mathbb{AR}^3$ by making the plane $w = 0$ the plane at infinity. The w-coordinate component of $\mathbf{p}(u)$ is

$w(u) = 1 - bu^2$, which has two zeros $u_1 = 1/\sqrt{b} \in I_1$ and $u_2 = -1/\sqrt{b} \in I_2$. The two points on the component P_1 of the QSIC corresponding to u_1 are

$$\mathbf{q}_0(u_1) = \begin{pmatrix} x_0(u_1) \\ y_0(u_1) \\ z_0(u_1) \\ w_0(u_1) \end{pmatrix} = \begin{pmatrix} bu_1^3 - u_1 \\ -(au_1 + \sqrt{s(u_1)}) \\ -u_1(au_1 + \sqrt{s(u_1)}) \\ 1 - bu_1^2 \end{pmatrix},$$

and

$$\mathbf{q}_1(u_1) = \begin{pmatrix} x_1(u_1) \\ y_1(u_1) \\ z_1(u_1) \\ w_1(u_1) \end{pmatrix} = \begin{pmatrix} bu_1^3 - u_1 \\ -(au_1 - \sqrt{s(u_1)}) \\ -u_1(au_1 - \sqrt{s(u_1)}) \\ 1 - bu_1^2 \end{pmatrix}.$$

There are now three subcases to be considered: (i) $a = 0$; (ii) $a > 0$; and (iii) $a < 0$.

First suppose $a = 0$. In this case,

$$\begin{aligned} s(u) &= -bu^4 + (b^2+1)u^2 - b \\ &= (u^2 - b)(1 - bu^2). \end{aligned}$$

Thus,

$$\mathbf{q}_0(u) = \sqrt{|1-bu^2|} \begin{pmatrix} -u\sqrt{|1-bu^2|} \\ -\sqrt{|u^2-b|} \\ -u\sqrt{|u^2-b|} \\ \sqrt{|1-bu^2|} \end{pmatrix},$$

$$\mathbf{q}_1(u) = \sqrt{|1-bu^2|} \begin{pmatrix} -u\sqrt{|1-bu^2|} \\ \sqrt{|u^2-b|} \\ u\sqrt{|u^2-b|} \\ \sqrt{|1-bu^2|} \end{pmatrix}.$$

When $u \to u_1$, $y_0(u_1), z_0(u_1)$ are nonzero but $w_0(u_1)$ is zero, and $y_1(u_1), z_1(u_1)$ are nonzero but $w_1(u_1)$ is zero. So the two points $\mathbf{q}_0(u_1)$ and $\mathbf{q}_1(u_1)$ are both at the infinity and become the same point. But nearby the infinity point, the w-components of both $\mathbf{q}_0(u)$ and $\mathbf{q}_1(u)$ are positive, while their y-components and z-components have opposite signs. This means that $\mathbf{q}_0(u)$ and $\mathbf{q}_1(u)$ go to the different directions at the infinity point, leading to the component P_1 of the QSIC not touching the infinity point, but crossing over it. Hence, the component P_1 has exactly one point at infinity on it in $\mathbb{AR}^3$, which is $\mathbf{q}_0(u_1)$=$\mathbf{q}_1(u_1)$.

In the second subcase, we suppose $a > 0$. In this case, $\sqrt{s(u_1)} = a/\sqrt{b}$, hence $y_0(u_1) = -2a/\sqrt{b}$ and $z_0(u_1) = -2a/b$. Since not all of $x_0(u_1)$, $y_0(u_1)$ and $z_0(u_1)$ are zero but $w_0(u_1)$ is zero, $\mathbf{q}_0(u_1)$ is a point at infinity. Next we consider $\mathbf{q}_1(u_1)$. Denote $f(u) = \sqrt{s(u)}$. The Taylor expansion of $f(u)$ at $u = u_1$ is

$$\begin{aligned} f(u) = \frac{a}{\sqrt{b}} + f'(u_1)(u-u_1) + \frac{f''(u_1)}{2}(u-u_1)^2 + ... \\ + \frac{f^n(u_1)}{n!}(u-u_1)^n + R_{n+1}(u), \end{aligned}$$

where $R_{n+1}(u) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(u-u_1)^{n+1}$. Then

$$\begin{aligned} \lim_{u \to u_1} \mathbf{q}_1(u) &= \begin{pmatrix} bu^3 - u \\ -(au - f(u)) \\ -u(au - f(u)) \\ 1 - bu^2 \end{pmatrix} \\ &= \begin{pmatrix} bu(u+\frac{1}{\sqrt{b}})(u-\frac{1}{\sqrt{b}}) \\ -(\frac{a}{\sqrt{b}} - \frac{a}{\sqrt{b}} - f'(u_1)(u-\frac{1}{\sqrt{b}})) \\ -u(\frac{a}{\sqrt{b}} - \frac{a}{\sqrt{b}} - f'(u_1)(u-\frac{1}{\sqrt{b}})) \\ -b(u+\frac{1}{\sqrt{b}})(u-\frac{1}{\sqrt{b}}) \end{pmatrix} \\ &= \begin{pmatrix} bu(u+\frac{1}{\sqrt{b}})(u-\frac{1}{\sqrt{b}}) \\ f'(u_1)(u-\frac{1}{\sqrt{b}}) \\ uf'(u_1)(u-\frac{1}{\sqrt{b}}) \\ -b(u+\frac{1}{\sqrt{b}})(u-\frac{1}{\sqrt{b}}) \end{pmatrix}. \end{aligned}$$

After removing the common factor $(u - \frac{1}{\sqrt{b}})$, we get

$$\begin{aligned} \lim_{u \to u_1} \mathbf{q}_1(u) &= \begin{pmatrix} bu(u+\frac{1}{\sqrt{b}}) \\ f'(u_1) \\ uf'(u_1) \\ -b(u+\frac{1}{\sqrt{b}}) \end{pmatrix} \\ &= \begin{pmatrix} 2, f'(u_1), \frac{f'(u_1)}{\sqrt{b}}, -2\sqrt{b} \end{pmatrix}. \end{aligned}$$

Hence $\mathbf{q}_1(u_1)$ is a finite point in $\mathbb{AR}^3$. In the third subcase of $a < 0$, it can be proved similarly that $\mathbf{q}_1(u_1)$ is a point at infinity and $\mathbf{q}_0(u_1)$ is a finite point in $\mathbb{AR}^3$. Hence, P_1 is a component of the QSIC that has exactly one point at infinity on it in $\mathbb{AR}^3$.

Now we show that P_1 is affinely infinite. For brevity, only the case of $a > 0$ is discussed here; similar arguments can be used in the other two cases $a < 0$ and $a = 0$. Since $x(u_1)$, $y(u_1)$, and $z(u_1)$ are not all zero and $u = u_1$ is a simple zero of $w(u)$, $\mathbf{q}_0(u)$ traverses the plane at infinity at $\mathbf{q}_0(u_1)$. Thus, given a plane L that does not contain the point at infinity on P_1, there is a $\delta > 0$ such that $\mathbf{q}_0(u_1 - \delta)$, $\mathbf{q}_0(u_1 + \delta)$ are on different sides of L. Then the continuous curve segment which comprises three curve segments
$C_1 : \mathbf{q}_1(u), u \in (u_-, u_+)$,
$C_2 : \mathbf{q}_0(u), u \in (u_-, u_1 - \delta)$,
and $C_3 : \mathbf{q}_0(u), u \in (u_1 + \delta, u_+)$,
must intersect the plane L at a finite point. Meanwhile, any other plane, including the plane at infinity of $\mathbb{AR}^3$ contains the point at infinity on P_1. Thus we conclude that any plane in $\mathbb{PR}^3$ intersects the component P_1, i.e., the component P_1 of the QSIC is affinely infinite. Similarly, it can be shown that the other component P_2 of the QSIC is also affinely infinite. Hence, the QSIC of $\mathcal{A}$ and $\mathcal{B}$ has two affinely infinite components. This completes the proof of Theorem 4. An example in this case is shown as Figure 6.

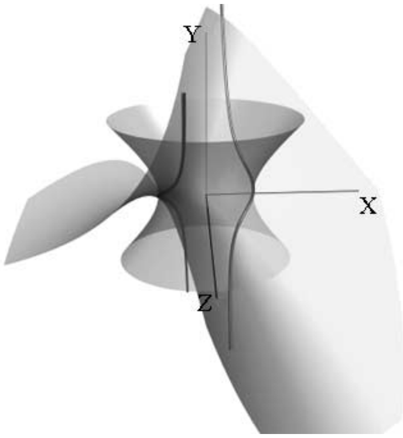

Figure 6. The case of the QSIC has two affinely infinite components.

Since Theorems 2 to 4 cover three different conditions based on the exhaustive enumeration of the number of real roots that a quartic equation with real coefficients can have and these conditions are sufficient for different morphologies of the QSIC, we conclude that these conditions are also necessary. This is summarized below in Theorem 5, which is the main contribution of the present paper.

Theorem 5: *Let $f(\lambda) = 0$ be the characteristic equation of two quadrics $\mathcal{A} : X^T AX = 0$ and $\mathcal{B} : X^T BX = 0$ with a nonsingular QSIC.*

1. *The $QSIC$ has either two affinely finite connected components or no real points in $\mathbb{PR}^3$ if and only if $f(\lambda) = 0$ has four distinct real roots.*
2. *The $QSIC$ has one affinely finite connected component in $\mathbb{PR}^3$ if and only if $f(\lambda) = 0$ has two distinct real roots and a pair of complex conjugate roots.*
3. *The $QSIC$ has two affinely infinite connected components in $\mathbb{PR}^3$ if and only if $f(\lambda) = 0$ has two distinct pairs of complex conjugate roots.*

4. Conclusion

We have presented necessary and sufficient conditions for classifying the morphology of the nonsingular intersection curve of two quadric surfaces (QSIC) in $\mathbb{PR}^3$ by considering the number of real roots of the characteristic equation of the two quadrics. This work is complementary to the work in [3] which classifies singular (or degenerate) QSICs based on the factorization of a planar quartic curve. We note that the classic results on the classification of QSICs in complex projective space using the Segre characteristic over the complex field cannot be translated to solve the present problem of classifying real nonsingular QSICs, since all topologically different real nonsingular QSICs give rise to the same Segre characteristic corresponding to that the characteristic equation has four distinct roots. So only by considering the number of the real roots of the characteristic equation of two quadrics in real projective space, we succeeded in classifying the morphology of a nonsingular QSIC in $\mathbb{PR}^3$. Thus our work can also be regarded as a specialization of the work in [1] from the complex space to the real space for nonsingular QSICs.

The condition given in Theorem 2 for the case where the characteristic equation has four distinct roots cannot completely resolve the classification of two important topologically different cases: the QSIC has two affinely finite connected components in $\mathbb{PR}^3$ or the QSIC has no real points. The following examples show that these two cases do arise. Consider the two quadrics defined by matrices $A = \mathrm{diag}(1, 1, 1, -1)$ and $B = \mathrm{diag}(1.10, 0.56, 0.12, -0.28)$, whose intersection curve has two components in $\mathbb{PR}^3$ (see the right figure in Figure 7). Then consider the two quadrics defined by matrices $A = \mathrm{diag}(1, 1, 1, -1)$ and $B = \mathrm{diag}(6.25, 5.44, 3.69, -7.69)$, which do not intersect in $\mathbb{PR}^3$ (see the left figure in Figure 7). Clearly, in both cases, the characteristic equation has four distinct real roots.

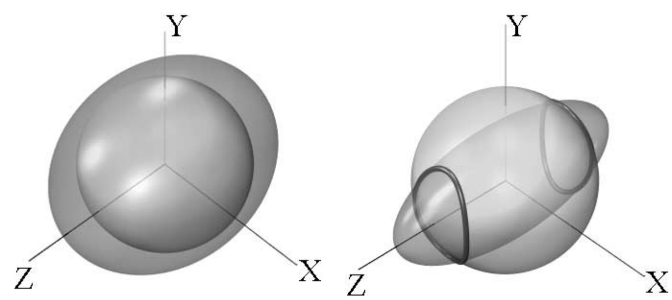

Figure 7. Two cases of the characteristic equation having four distinct roots.

However, with a computational approach, these two cases above can be distinguished as follows. First solve for the four real roots λ_i of the characteristic equation $\mathrm{f}(\lambda) = 0$, and then compute their associate real eigenvectors V_i by solving the equations $(\lambda_i A - B)X = 0$, $i = 1, 2, 3, 4$. The matrix $V = (V_1, V_2, V_3, V_4)$ can be used in a congruent transformation to reduce both A and B into diagonal forms, representing two quadrics $\mathcal{A}'$ and $\mathcal{B}'$ in canonical forms. What remains is a simple task of testing whether $\mathcal{A}'$ and $\mathcal{B}'$ intersect or not in $\mathbb{PR}^3$. Note that $\mathcal{A}'$ and $\mathcal{B}'$ intersect if and only if the original quadrics $\mathcal{A}$ and $\mathcal{B}$ intersect in $\mathbb{PR}^3$. In this case, if $\mathcal{A}$ and $\mathcal{B}$ intersect in $\mathbb{PR}^3$, then their QSIC has two affinely finite connected components.

Further research is needed to gain a better understanding of quadric pencils in real space. It has proven fruitful to study on the relationship between the roots of the characteristic equation and the configurations of two quadrics or their QSIC. One of the open problems is to find a more refined algebraic condition for distinguishing the two morphologies of a nonsingular QSIC covered in Theorem 2. Another issue is computing the simultaneous block diagonalization forms to obtain a parametric equation of the QSIC, guided by the topological information obtained using the results of this paper.

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